Inequality involving triangles

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Let x,y,z be positive real numbers. Then in triangle ABC with semiperimeter s and inradius r

$$\frac{x}{y+z}\cot^2\frac{A}{2}+\frac{y}{z+x}\cot^2\frac{B}{2}+\frac{z}{x+y}\cot^2\frac{C}{2}\geq 18-\frac{s^2}{2r^2}$$

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Since $\cot\frac{A}{2}=\frac{s-a}{r}$ then $\sum\frac{x}{y+z}\cot^2\frac{A}{2}\geq 18-\frac{s^2}{2r^2}\Leftrightarrow$

$$\sum\frac{x}{y+z}\cdot\frac{(s-a)^2}{r^2}\cot^2\frac{A}{2}\geq 18-\frac{s^2}{2r^2}\Leftrightarrow$$

(**1**) $\quad\sum\frac{x}{y+z}(s-a)^2\geq 18r^2-\frac{s^2}{2}.$

$\sum(s-a)^2=(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2-2s(a+b+c)+3s^2=$
$a^2+b^2+c^2-s^2=2(s^2-4Rr-r^2)-s^2=s^2-2r^2-8Rr.$

In particular for $x=y=z=1$ we obtain

$\frac{1}{2}\sum(s-a)^2-\left(18r^2-\frac{s^2}{2}\right)=\frac{s^2}{2}-r^2-4Rr-\left(18r^2-\frac{s^2}{2}\right)=s^2-19r^2-4Rr\geq$
$16Rr-5r^2-19r^2-4Rr=12r(R-2r)$

Lemma. For any positive real numbers $a,b,c,x,y,z>0$ holds inequality

(**2**) $\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\geq\frac{\Delta(a,b,c)}{2}$, where
$\Delta(a,b,c):=2ab+2bc+2ca-a^2-b^2-c^2$ [1], [2].

Equality occurs iff $b+c-a,c+a-b,a+b-c>0$ and
$(x,y,z)=k(b+c-a,c+a-b,a+b-c)$
for some positive k.

Proof.

Let $S:=\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2$

By Cauchy Inequality

$$S+a^2+b^2+c^2=(x+y+z)\sum\frac{a^2}{y+z}\geq(x+y+z)\cdot\frac{(a+b+c)^2}{2(x+y+z)}=\frac{(a+b+c)^2}{2}.$$

Hence, $S\geq\frac{(a+b+c)^2}{2}-(a^2+b^2+c^2)=\frac{\Delta(a,b,c)}{2}$ and equality occurs iff
$(y+z,z+x,x+y)=k(a,b,c)\Leftrightarrow(x,y,z)=k(b+c-a,c+a-b,a+b-c)$,where $k>0$.■

Coming back to tour problem, by replacing (a,b,c) in (**2**) with $(s-a,s-b,s-c)$ we obtain

$$\sum\frac{x}{y+z}(s-a)^2\geq\frac{\Delta(s-a,s-b,s-c)}{2}.$$

Thus remains to prove

$$\frac{\Delta(s-a,s-b,s-c)}{2}\geq 18r^2-\frac{s^2}{2}\Leftrightarrow\Delta(s-a,s-b,s-c)\geq 36r^2-s^2.$$

Let $x:=s-a,y:=s-b,z:=s-c$. Then, using normalization $s=1$ (due to homogeneity of the latter inequality) and denoting $p:=xy+yz+zx,q:=xyz$ we obtain

$\Delta(s-a,s-b,s-c)=$

$2(xy+yz+zx)-(x^2+y^2+z^2)=4p-1,r^2=\frac{(s-a)(s-b)(s-c)}{s}=q.$

Thus,we have $\Delta(s-a,s-b,s-c)-(36r^2-s^2)=4p-1-(36q-1)=4(p-9q)\geq 0$

because $p = (xy + yz + zx)(x + y + z) \geq 3\sqrt[3]{x^2y^2z^2} \cdot 3\sqrt[3]{xyz} = 9q.$

Remark.

More inequalities with vanished variables and involving symmetric polynomial $\Delta(a,b,c)$ in

1. A.Alt- Variations on an algebraic inequality-MR n.2, 2009

Link:

http://equationroom.com/Publications/Mathematical%20Reflections/Variations%20on%20an%20algebraic%20inequality-n.2,%202009.pdf

or, https://www.academia.edu/32055493/Variations_on_an_algebraic_inequality

Also more information about polynomial $\Delta(a,b,c)$ and its application in

2.A.Alt -Geometric Inequalities with polynomial 2xy+2yz+2zx2-sqr(x)-sqr(y)-sqr(z), Octogon v.22,n.2- 2014

Link:

http://www.equationroom.com/Publications/Mathematical%20Olympiads%20Group/Geometric%20Inequalities%20with%20polynomial%202xy+2yz+2zx2-sqr(x)-sqr(y)-sqr(z),

or,

https://www.academia.edu/32055494/Geometric_Inequalities_with_polynomial_2xy_2yz_2zx Octogon_Mathematical_Magazine_v.22_n.2-_2014.pdf